

# Dispersive effective equations for waves in heterogeneous media on large time scales

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February 21, 2013

## Abstract

We investigate second order linear wave equations in periodic media, aiming at the derivation of effective equations in  $\mathbb{R}^n$ ,  $n \geq 1$ . Standard homogenization theory provides, for the limit of a small periodicity length  $\varepsilon > 0$ , an effective second order wave equation that describes solutions on time intervals  $[0, T]$ . In order to approximate solutions on large time intervals  $[0, T\varepsilon^{-2}]$ , one has to use a dispersive, higher order wave equation. In this work, we provide a well-posed, weakly dispersive effective equation, and an estimate for errors between the solution of the original heterogeneous problem and the solution of the dispersive wave equation. We use Bloch-wave analysis to identify a family of relevant limit models and introduce an approach to select a well-posed effective model. The analytical results are confirmed and illustrated by numerical tests.

**Keywords:** homogenization, wave equation, weakly dispersive model, Bloch-wave expansion

**MSC:** 35B27, 35L05

## 1 Introduction

The wave equation describes wave propagation in very different applications, ranging from elastic waves to electro-magnetic waves. In the two mentioned applications, it is of interest to describe waves in periodic media where the period is much smaller than the lengthscale of the wave. The most fundamental questions regard the effective wave speed and a possible dispersive behavior due to heterogeneities.

We concentrate on the simplest model, the second order wave equation in divergence form. For notational convenience we omit here a positive coefficient in the acceleration term and study, for  $x \in \mathbb{R}^n$ , the wave equation

$$\partial_t^2 u^\varepsilon(x, t) = \nabla \cdot (a^\varepsilon(x) \nabla u^\varepsilon(x, t)). \quad (1.1)$$

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The medium is characterized by the coefficient function  $a^\varepsilon : \mathbb{R}^n \rightarrow (0, \infty)$ . Since we are interested in periodic media with a small periodicity length-scale  $\varepsilon > 0$ , we assume that  $a^\varepsilon(x) = a_Y(x/\varepsilon)$ , where  $a_Y : \mathbb{R}^n \rightarrow \mathbb{R}$  has the periodicity of the cube  $Y := (-\pi, \pi)^n \subset \mathbb{R}^n$ . The wave equation is complemented with the initial condition

$$u^\varepsilon(x, 0) = f(x), \quad \partial_t u^\varepsilon(x, 0) = 0. \quad (1.2)$$

We consider initial data  $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$  of the form

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} F_0(k) e^{+ik \cdot x} dk, \quad (1.3)$$

and assume that the (bounded) function  $F_0 : \mathbb{R}^n \rightarrow \mathbb{C}$  has a compact support  $K \subset \subset \mathbb{R}^n$ . In this sense,  $f$  has a finite support in Fourier space.

The fundamental question of homogenization theory is: For small  $\varepsilon > 0$ , can the solution  $u^\varepsilon$  be approximated by a solution of an equation with constant coefficients? The answer is affirmative: There exists an effective coefficient matrix  $A \in \mathbb{R}^{n \times n}$ , computable from  $a_Y$ , such that the following holds: on an arbitrary time interval  $[0, T]$ , if we define  $w : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  as the solution of

$$\partial_t^2 w(x, t) = \nabla \cdot (A \nabla w(x, t)), \quad w(x, 0) = f(x), \quad \partial_t w(x, 0) = 0, \quad (1.4)$$

there holds  $u^\varepsilon \rightarrow w$  as  $\varepsilon \rightarrow 0$ . For the result and function spaces see e.g. [4].

We are interested in a refinement of this result. Our aim is to investigate the behavior of solutions  $u^\varepsilon$  of (1.1) for large times. To formulate a precise result, we start from a positive number  $T_0 > 0$ , and investigate solutions for all  $t \in [0, T_0 \varepsilon^{-2}]$ . It is well-known that the homogenized equation (1.4) cannot provide an approximation of  $u^\varepsilon$  on the interval  $[0, T_0 \varepsilon^{-2}]$ . Instead, we need a dispersive equation to approximate  $u^\varepsilon$ .

**Main result.** In addition to the coefficient matrix  $A$ , we will define matrices  $E$  and  $F$ . All matrices are computable from the coefficient  $a_Y(\cdot)$  with the help of an eigenvalue problem on the periodicity cell  $Y$ . The constant coefficient matrices define linear spatial differential operators, the two second order operators  $AD^2 = \sum_{i,j} A_{ij} \partial_i \partial_j$  and  $ED^2 = \sum_{i,j} E_{ij} \partial_i \partial_j$ , and the fourth order operator  $FD^4 = \sum_{i,j,m,l} F_{ijml} \partial_i \partial_j \partial_m \partial_l$ . We formulate a weakly dispersive equation

$$\partial_t^2 w^\varepsilon = AD^2 w^\varepsilon + \varepsilon^2 ED^2 \partial_t^2 w^\varepsilon - \varepsilon^2 FD^4 w^\varepsilon. \quad (1.5)$$

As initial conditions we use once more  $w^\varepsilon(x, 0) = f(x)$  and  $\partial_t w^\varepsilon(x, 0) = 0$ . Equation (1.5) is of fourth order in the spatial variables, but it contains additionally an operator which involves second spatial and second time derivatives. The operator contains the small parameter  $\varepsilon > 0$  explicitly; formally, for  $\varepsilon = 0$ , we recover the homogenized equation (1.4).

Our main result shows that the weakly dispersive equation (1.5) provides, at the lowest order, an approximation of the original equation for large times.

**Theorem 1.1.** *Let  $\varepsilon = \varepsilon_l \rightarrow 0$  be a sequence of positive numbers,  $n \geq 1$  the dimension. Let the heterogeneous medium be given by a  $Y$ -periodic positive function  $a_Y : \mathbb{R}^n \rightarrow \mathbb{R}$  of class  $C^2$ , symmetric under reflections  $x_j \longleftrightarrow -x_j$ , and symmetric under coordinate exchanges  $x_j \longleftrightarrow x_k$ , see (2.23). Let initial data be given by  $f \in L^1(\mathbb{R}^n)$ , such that the Fourier transform  $F_0$  has compact support, see (1.3). We assume that the Bloch-expansion (2.8) of the initial data  $f$  is convergent in  $H^1(\mathbb{R}^n)$ .*

*We use the coefficient matrices  $A$  and  $C$  defined in (2.19), and  $E$  and  $F$  as defined in Lemma 3.1. Then the following holds:*

1. **Well-posedness** Equation (1.5) with initial condition (1.2) has a unique solution  $w^\varepsilon$  for all positive times (see Theorem 3.1 for function spaces).
2. **Error estimate** Let  $w^\varepsilon$  be the solution of (1.5), and let  $u^\varepsilon$  be the solution of (1.1) for the same initial condition (1.2). Then, with a constant  $C_0 = C_0(a_Y, T_0, f)$ ,

$$\sup_{t \in [0, T_0 \varepsilon^{-2}]} \|u^\varepsilon - w^\varepsilon\|_{L^2(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)} \leq C_0 \varepsilon. \quad (1.6)$$

The definition of the norm in (1.6) is recalled at the end of Section 2.2. We note that  $f \in L^1(\mathbb{R}^n)$  implies  $F_0 \in L^\infty(\mathbb{R}^n)$ . Since  $F_0$  has compact support, we obtain  $F_0 \in L^2(\mathbb{R}^n)$ ,  $f \in L^2(\mathbb{R}^n)$ , and the regularity  $f \in H^s(\mathbb{R}^n)$  for every  $s \geq 0$ . From now on, we understand the Fourier transform (1.3) in the sense of  $L^2(\mathbb{R}^n)$ .

## Comparison with the literature

The derivation of effective equations in periodic homogenization problems is an old subject [22], two-scale convergence [1] is today the most relevant analytical tool. The use of Bloch-wave expansions was explored only more recently, see e.g. [7].

Compared to elliptic and parabolic equations, some distinctive features are relevant in the analysis of the wave equation. One observation of [4] was that convergence of energies can only be expected for initial data that are adapted to the periodic medium, see also [14]. Diffraction and dispersion effects are analyzed in the spirit of homogenization theory in [2, 3]. While the underlying questions are similar, these contributions study a different scaling behavior in  $\varepsilon$ . Other homogenization results for the wave equation are contained in [5, 15, 19, 20, 24, 25].

The interest in the derivation of a dispersive effective wave equation appears most clearly in the works of Chen, Fish, and Nagai, e.g. [10, 11, 12, 13]. The authors expand several ideas to treat the problem, among others they propose to introduce a slow and a fast time scale to capture the long-time behavior of waves. The authors concentrate on numerical studies and do not provide a derivation of an effective model.

**Derivation of dispersive models.** To our knowledge, the first rigorous result that establishes a dispersive model for the wave equation in the scaling of (1.1)

appeared in [18]. In that contribution, the one-dimensional case was analyzed, the one-dimensional version of (1.5) was formulated (in this case,  $A$ ,  $E$ , and  $F$  are scalar coefficients and the differential operator is  $D = \partial_x$ ), and a result similar to our Theorem 1.1 was shown: the well-posedness of the dispersive equation and an error bound on large time intervals.

Beyond the one dimensional case, we are not aware of any rigorous results. The most relevant contribution with the perspective taken here is [23]. In that paper, Bloch-wave expansions are used to analyze the problem, mathematical insight is gained, and the dispersive wave equation (3.1) is formulated (not in one of the theorems, but as a formal consequence on page 992). We use many of the ideas of that contribution.

Equation (3.1) appears also as equation (42) in [13], the authors call it the “bad” Boussinesq equation. The problem about this equation is its ill-posedness: Loosely speaking, the equation is of the form  $\partial_t^2 u + Lu = 0$ , with  $L = -\Delta - \varepsilon \Delta^2$ . The lowest order part (in  $\varepsilon$ ) of  $L$  is  $-\Delta$ , hence a positive operator, but for every  $\varepsilon > 0$ , the operator is negative, since  $\Delta^2$  is positive and contains the highest order of differentiation. One can speculate that this was the reason why no effective dispersive models were rigorously formulated in the above mentioned works.

It was already observed in [13], that a “good” Boussinesq equation can be obtained with a simple trick: Loosely speaking, in an equation of the form  $\partial_t^2 u = -Lu = \Delta u + \varepsilon \Delta^2 u$ , we can replace  $\Delta u$  to lowest order (in  $\varepsilon$ ) by  $\partial_t^2 u$ , hence we may write the equation as  $\partial_t^2 u = \Delta u + \varepsilon \Delta \partial_t^2 u$ . In this form, the equation is well-posed. This observation was also exploited in [18], where it was shown rigorously, that the “good” Boussinesq equation is the effective model for large times in the one-dimensional case.

In this contribution we treat the higher dimensional case. The methods are completely different from those of [18], but are based on Bloch-wave expansions as used in [23]. We must introduce two assumptions: (i) we use initial data that are compactly supported in Fourier space and (ii) we assume some symmetry of the medium. Under these assumptions, we derive an effective dispersive equation. Regarding the replacement that transforms the “bad” effective equation into a “good” one, we must work with tensors of coefficients, but the essential idea remains the same. We show with mathematical rigor that the new equation has the desired approximation property.

In Section 3 we expand the solution  $u^\varepsilon$  in Bloch waves, in Section 3 we analyze the weakly dispersive equation (1.5). The proof of Theorem 1.1 is concluded at the end of Section 3.

## 2 Approximation with a Bloch wave expansion

In this section we present, in slightly changed notation and with mathematical rigor regarding assumptions and norms, the approximation results of [23]. To simplify some of the notation of [23], we consider here only the mass-density  $\bar{\rho} \equiv 1$  and the scaling factor  $\lambda = 1$ . We abbreviate the square of certain eigenvalues with  $\mu := \bar{\omega}^2$ .

## 2.1 Bloch wave expansion

We are given a medium by the coefficient  $a_Y(y)$  on the cube  $Y$ . The Bloch wave expansion uses functions  $\psi_m$ , which are solutions of a periodic eigenvalue problem on  $Y$ . The wave parameter  $k$  is a vector in the reciprocal periodicity cell  $Z = (-1/2, 1/2)^n$ . At this point, we regard  $k \in Z$  as a given parameter and consider

$$-(\nabla_y + ik) \cdot (a_Y(y)(\nabla_y + ik)\psi_m(y, k)) = \mu_m(k)\psi_m(y, k). \quad (2.1)$$

We search for  $\psi_m(\cdot, k) : Y \rightarrow \mathbb{C}$  in the space  $H_{\text{per}}^1(Y)$ , defined as the space of periodic functions on  $Y$  of class  $H^1$ . We find a family (indexed by  $m \in \mathbb{N} = \{0, 1, 2, \dots\}$ ) of periodic solutions  $\psi_m(\cdot, k) : Y \rightarrow \mathbb{C}$  with eigenvalues  $\mu_m(k)$ , both the solution and the eigenvalue depend on  $k$ . We assume that the functions are normalized in  $L^2(Y)$ ,  $\|\psi_m\|_{L^2(Y)} = 1$ . Regarding the regularity of  $\psi_m$  we note that, for  $a_Y$  of class  $C^1$ , standard elliptic regularity theory implies  $\psi_m \in H^2(Y)$ .

Based on the eigenfunction  $\psi_m$ , we can construct the quasi-periodic function  $w_m(y, k) := \psi_m(y, k)e^{ik \cdot y}$ . The eigenvalue problem (2.1) is designed such that  $w_m$  has the eigenfunction property

$$-\nabla_y \cdot (a_Y(y)\nabla_y w_m(y, k)) = \mu_m(k)w_m(y, k). \quad (2.2)$$

We recall an essential fact regarding the completeness of these eigenfunctions (see e.g. [7] for this well-known result). The Bloch solutions form a basis of  $L^2(\mathbb{R}^n)$  in the sense that every function  $g \in L^2(\mathbb{R}^n)$  can be expanded as

$$g(y) = \sum_{m=0}^{\infty} \int_Z \hat{g}_m(k) w_m(y, k) dk, \quad \hat{g}_m(k) = \int_{\mathbb{R}^n} g(y) w_m(y, k)^* dy, \quad (2.3)$$

where we use the star  $*$  to denote complex conjugation. There holds the Parseval identity

$$\|g\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |g(x)|^2 dx = \sum_{m=0}^{\infty} \int_Z |\hat{g}_m(k)|^2 dk = \|\hat{g}\|_{l^2(\mathbb{N}, L^2(Z))}^2. \quad (2.4)$$

### Rescaled Bloch wave expansion

We investigate a strongly heterogeneous medium  $a^\varepsilon(x) = a_Y(x/\varepsilon)$ . Starting from the Bloch waves on the cube  $Y$ , we therefore define rescaled quantities as

$$\psi_m^\varepsilon(x, k) := \psi_m\left(\frac{x}{\varepsilon}, \varepsilon k\right), \quad \mu_m^\varepsilon(k) := \frac{1}{\varepsilon^2} \mu_m(\varepsilon k), \quad (2.5)$$

$$w_m^\varepsilon(x, k) := w_m\left(\frac{x}{\varepsilon}, \varepsilon k\right) = \psi_m^\varepsilon(x, k) e^{ik \cdot x} = \psi_m\left(\frac{x}{\varepsilon}, \varepsilon k\right) e^{ik \cdot x}. \quad (2.6)$$

This choice guarantees, in particular,

$$-\nabla \cdot (a^\varepsilon(x)\nabla w_m^\varepsilon(x, k)) = \mu_m^\varepsilon(k)w_m^\varepsilon(x, k). \quad (2.7)$$

The expansion formula (2.3) in Bloch eigenfunctions can be expressed in the new variables. Every function  $f \in L^2(\mathbb{R}^n)$  can be written as

$$f(x) = \sum_{m=0}^{\infty} \int_{Z/\varepsilon} \hat{f}_m^\varepsilon(k) w_m^\varepsilon(x, k) dk, \quad \hat{f}_m^\varepsilon(k) = \int_{\mathbb{R}^n} f(x) w_m^\varepsilon(x, k)^* dx. \quad (2.8)$$

To verify this formula, it suffices to set  $f(x) = g(x/\varepsilon)$  and  $\hat{f}_m^\varepsilon(k) = \varepsilon^n \hat{g}_m(\varepsilon k)$ . This shows additionally the Parseval identity in transformed variables,

$$\|f\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |f(x)|^2 dx = \sum_{m=0}^{\infty} \int_{Z/\varepsilon} |\hat{f}_m^\varepsilon(k)|^2 dk = \|\hat{f}^\varepsilon\|_{l^2(\mathbb{N}, L^2(Z/\varepsilon))}^2. \quad (2.9)$$

### Expansion of the solution

The Bloch-wave formalism can provide a formula for the solution.

**Lemma 2.1** (Expansion of the solution). *Let the  $Y$ -periodic coefficients be of class  $a_Y \in C^1(\mathbb{R}^n)$  and positive, let initial data  $f \in H^1(\mathbb{R}^n)$  be given as in (1.3). Then, for every  $\varepsilon > 0$ , the wave equation (1.1) has a unique weak solution  $u^\varepsilon$  with the regularity  $u^\varepsilon(x, t) \in L^\infty([0, \infty), H^2(\mathbb{R}^n)) \cap W^{1,\infty}([0, \infty), H^1(\mathbb{R}^n))$ .*

*We expand the initial values  $f$  as in (2.8) and make the assumption that the series is convergent in  $H^1(\mathbb{R}^n)$ . Then the weak solution  $u^\varepsilon$  of (1.1) can be represented as*

$$u^\varepsilon(x, t) = \sum_{m=0}^{\infty} \int_{Z/\varepsilon} \hat{f}_m^\varepsilon(k) w_m^\varepsilon(x, k) \operatorname{Re} \left( e^{it\sqrt{\mu_m^\varepsilon(k)}} \right) dk. \quad (2.10)$$

*The right hand side is understood as the strong  $L^2(\mathbb{R}^n)$  limit of partial sums, for every fixed  $t \geq 0$ .  $\operatorname{Re}(\cdot)$  denotes the real part.*

Before we start the proof, we note that the expression in (2.10) formally defines a solution of (1.1)–(1.3). In fact, the second time derivative of the right hand side is given by the same formula, introducing only the additional factor  $-\mu_m^\varepsilon(k)$  under the integral. On the other hand, the application of the operator  $\nabla \cdot (a^\varepsilon(x) \nabla)$  to the integrand produces, by (2.7), the same result.

*Proof. Step 1. The energy solution.* The solution  $u^\varepsilon$  can be constructed, e.g., with a Galerkin scheme. One exploits the energy estimate which is obtained with a multiplication of the equation with (the real function)  $\partial_t u^\varepsilon$ ,

$$0 = \int_{\mathbb{R}^n} [\partial_t^2 u^\varepsilon(\cdot, t) - \nabla \cdot (a^\varepsilon \nabla u^\varepsilon(\cdot, t))] \partial_t u^\varepsilon = \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^n} |\partial_t u^\varepsilon|^2 + a^\varepsilon |\nabla u^\varepsilon(\cdot, t)|^2.$$

Also higher order estimates can be obtained. We use  $L^\varepsilon := \nabla \cdot (a^\varepsilon(x) \nabla)$  and multiply the equation  $\partial_t^2 u^\varepsilon = L^\varepsilon u^\varepsilon$  with  $-\partial_t (L^\varepsilon u^\varepsilon)$  to find

$$\frac{d}{dt} \frac{1}{2} \int a^\varepsilon |\partial_t \nabla u^\varepsilon|^2 + |L^\varepsilon u^\varepsilon|^2 = 0. \quad (2.11)$$

Since the initial data are  $u(t=0) = f \in H^2(\mathbb{R}^n)$  and  $\partial_t u(t=0) \equiv 0$ , we obtain estimates for  $u^\varepsilon$  in the stated function spaces. Estimates for  $L^\varepsilon u^\varepsilon \in L^2$  imply

estimates for  $u^\varepsilon \in H^2$  due to  $a_Y \in C^1$  by standard elliptic regularity theory. Uniqueness within the given class follows from linearity and a similar calculation.

*Step 2. Convergence in (2.10).* The Parseval identity (2.9) implies that the coefficient functions define an element  $(\hat{f}_m^\varepsilon(k))_{m,k}$  of  $l^2(\mathbb{N}, L^2(Z/\varepsilon))$ . As a consequence, also the modified coefficients  $\left(\hat{f}_m^\varepsilon(k) \operatorname{Re}\left(e^{it\sqrt{\mu_m^\varepsilon(k)}}\right)\right)_{m,k}$  define an element in the same space, since all factors have the unit norm. Using again the Parseval identity (2.9), we conclude that the sum of (2.10) converges in  $L^2(\mathbb{R}^n)$ , independent of  $t \geq 0$ .

*Step 3. Identification of  $u^\varepsilon$ .* We consider a partial sum  $\sum_{m=1}^M$  in (2.10) to define a function  $u_M^\varepsilon$  and observe that this provides a strong solution  $u_M^\varepsilon$  of the wave equation to the initial values  $f_M = \sum_{m=0}^M \int_{Z/\varepsilon} \hat{f}_m^\varepsilon(k) w_m^\varepsilon(x, k) dk$  and vanishing initial velocity. This fact can be checked with the above-mentioned formal calculation, the operator  $\nabla \cdot (a^\varepsilon(x) \nabla)$  is understood in the weak form and can be applied to the  $H^1(Y)$ -functions  $w_m$ . We claim that  $u_M^\varepsilon$  forms a Cauchy sequence in every space  $L^\infty([0, T], H^1(\mathbb{R}^n))$ . This follows with a testing argument, exploiting

$$\int_{\mathbb{R}^n} a^\varepsilon |\nabla u_M^\varepsilon(t) - \nabla u_N^\varepsilon(t)|^2 + |\partial_t u_M^\varepsilon(t) - \partial_t u_N^\varepsilon(t)|^2 = \int_{\mathbb{R}^n} a^\varepsilon |\nabla f_M^\varepsilon - \nabla f_N^\varepsilon|^2 \rightarrow 0$$

for  $M, N \rightarrow \infty$  due to the assumed  $H^1$ -convergence of (2.8). We conclude that  $u_M^\varepsilon$  converges to a limit function. The limit function is again an energy solution of the wave equation, by uniqueness we conclude  $u_M^\varepsilon \rightarrow u^\varepsilon$  for  $M \rightarrow \infty$ .

On the other hand, as observed in Step 2, by definition of  $u_M^\varepsilon$ , the limit function is given by the right hand side of (2.10).  $\square$

## 2.2 The approximation results of Santosa and Symes

With the next two theorems we observe that, for small  $\varepsilon > 0$ , the expression of (2.10) may be simplified. In our first simplification we realize that all indices  $m$  with  $m \geq 1$  can be neglected.

**Theorem 2.1** (Santosa and Symes [23], Theorem 1). *Let  $a_Y$  be of class  $C^1$ , let the initial values be given by  $f \in H^2(\mathbb{R}^n)$  of the form (1.3), such that the series of (2.8) is convergent in  $H^1(\mathbb{R}^n)$ . Let  $u^\varepsilon \in L^\infty((0, \infty); H^2(\mathbb{R}^n))$  be the solution of (1.1), given by (2.10). Then there exists  $C = C(f) > 0$  such that*

$$\sup_{t \in (0, \infty)} \left\| \sum_{m=1}^{\infty} \int_{Z/\varepsilon} \hat{f}_m^\varepsilon(k) w_m^\varepsilon(x, k) \operatorname{Re}\left(e^{it\sqrt{\mu_m^\varepsilon(k)}}\right) dk \right\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon. \quad (2.12)$$

*Proof.* We consider a single coefficient  $\hat{f}_m^\varepsilon(k) e^{\pm it\sqrt{\mu_m^\varepsilon(k)}}$  in the expansion of  $u^\varepsilon$  in (2.10). We use first the inversion formula (2.8) to express this coefficient, then the eigenvalue property (2.7) to introduce the factor  $\mu_m^\varepsilon(k)$ , then integration by



parts and the solution property of  $u^\varepsilon$ ,

$$\begin{aligned} \hat{f}_m^\varepsilon(k) \operatorname{Re} \left( e^{it\sqrt{\mu_m^\varepsilon(k)}} \right) &= \int_{\mathbb{R}^n} u^\varepsilon(x, t) w_m^\varepsilon(x, k)^* dx \\ &= -\frac{1}{\mu_m^\varepsilon(k)} \int_{\mathbb{R}^n} u^\varepsilon(x, t) [\nabla \cdot (a^\varepsilon(x) \nabla w_m^\varepsilon(x, k))]^* dx \\ &= -\frac{\varepsilon^2}{\mu_m(\varepsilon k)} \int_{\mathbb{R}^n} [\partial_t^2 u^\varepsilon(x, t)] w_m^\varepsilon(x, k)^* dx. \end{aligned} \quad (2.13)$$

We claim that, with  $C > 0$  independent of  $t \in [0, \infty)$ , the functions  $x \mapsto \partial_t^2 u^\varepsilon(x, t)$  satisfy the estimate  $\|\partial_t^2 u^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon^{-1}$ . Indeed, this bound can be obtained as in (2.11), where multiplication of  $\partial_t^2 u^\varepsilon = L^\varepsilon u^\varepsilon$  with  $\partial_t L^\varepsilon u^\varepsilon$  provided

$$\int_{\mathbb{R}^n} a^\varepsilon |\partial_t \nabla u^\varepsilon(\cdot, t)|^2 + |L^\varepsilon u^\varepsilon(\cdot, t)|^2 = \int_{\mathbb{R}^n} |L^\varepsilon u^\varepsilon(\cdot, 0)|^2.$$

Since initial data  $f$  are smooth, we have  $\|L^\varepsilon u^\varepsilon|_{t=0}\|_{L^2} = \|\nabla \cdot (a^\varepsilon(x) \nabla f)\|_{L^2} \leq C\varepsilon^{-1}$ , hence  $\|L^\varepsilon u^\varepsilon(\cdot, t)\|_{L^2} \leq C\varepsilon^{-1}$ . Accordingly, by the evolution equation, we also have  $\|\partial_t^2 u^\varepsilon(\cdot, t)\|_{L^2} = \|L^\varepsilon u^\varepsilon(\cdot, t)\|_{L^2} \leq C\varepsilon^{-1}$ .

We can now continue (2.13). Using again the Parseval identity, we conclude

$$\left\| \mu_m(\varepsilon k) \hat{f}_m^\varepsilon(k) \operatorname{Re} \left( e^{it\sqrt{\mu_m^\varepsilon(k)}} \right) \right\|_{l^2(\mathbb{N}, L^2(Z/\varepsilon))} = \varepsilon^2 \|\partial_t^2 u^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon.$$

It remains to observe that omitting the term  $m = 0$  decreases the norm on the left hand side. Regarding terms with  $m \geq 1$ , we exploit that there exists a lower bound  $c_0 > 0$  such that  $\mu_m(\xi) \geq c_0$ , independent of  $\xi \in Z$  and  $m \geq 1$ , cf. [7]. Another application of the Parseval identity provides the claim (2.12).  $\square$

At this point, we have obtained a first approximation of the solution. In the expansion of  $u^\varepsilon$ , all contributions from indices  $m \geq 1$  are not relevant at the lowest order (uniformly in time). Theorem 2.1 thus reads  $\|u^\varepsilon - u_0^\varepsilon\|_{L^\infty((0, \infty), L^2(\mathbb{R}^n))} \leq C\varepsilon$ , where

$$u_0^\varepsilon(x, t) := \int_{Z/\varepsilon} \hat{f}_0^\varepsilon(k) w_0^\varepsilon(x, k) \operatorname{Re} \left( e^{it\sqrt{\mu_0^\varepsilon(k)}} \right) dk. \quad (2.14)$$

Our next aim is to replace the Bloch coefficient  $\hat{f}_0^\varepsilon(k)$  by the Fourier coefficient  $F_0$ . At this point, we make more substantial changes with respect to [23]. Nevertheless, the essence of the subsequent result has been observed in Theorem 2 by Santosa and Symes, [23].

**Theorem 2.2.** *Let the dimension be  $n \geq 1$ , let the initial data  $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$  satisfy (1.3) with  $F_0 \in L^2(\mathbb{R}^n)$  supported on the compact set  $K$ . We assume that the coefficient  $a_Y$  has a regularity that implies  $\psi_0(\cdot, \xi) \in C^1(Y, \mathbb{R})$ , with a bound that is independent of  $\xi \in Z$ . Then, with  $C = C(f) > 0$ , there holds*

$$\left\| \hat{f}_0^\varepsilon - F_0 \right\|_{L^1(Z/\varepsilon)} \leq C\varepsilon. \quad (2.15)$$

Furthermore, for  $\varepsilon > 0$  small enough to satisfy  $\operatorname{diam}(K) < \varepsilon^{-1}$ , there holds

$$\hat{f}_0^\varepsilon(k) = 0 \quad \forall k \in (Z/\varepsilon) \setminus K. \quad (2.16)$$



We note that the regularity requirement on  $\psi_0$  is satisfied, e.g., for  $a_Y \in C^1$ , see the proof of the main theorem at the end of Section 3.

*Proof. Step 1:*  $k \in K$ . We write the difference of the two functions for fixed  $k$  as

$$\hat{f}_0^\varepsilon(k) - F_0(k) = \int_{\mathbb{R}^n} f(x) e^{-ik \cdot x} \left[ \psi_0 \left( \frac{x}{\varepsilon}, \varepsilon k \right)^* - \frac{1}{\sqrt{|Y|}} \right] dx.$$

The periodic solution  $\psi_0(\cdot, 0)$  to the wave vector  $k = 0$  is constant, by our normalization it is given as  $\psi_0(y, 0) = \sqrt{|Y|}^{-1}$  for every  $y \in Y$ . Since  $k$  ranges (in this step of the proof) in the bounded compact set  $K$ , we find the estimate

$$\left| \psi_0 \left( \frac{x}{\varepsilon}, \varepsilon k \right)^* - \frac{1}{\sqrt{|Y|}} \right| \leq C\varepsilon, \quad (2.17)$$

uniformly in  $k \in K$  and  $x \in \mathbb{R}^n$  for some constant  $C = C(a_Y)$ . This is a consequence of the fact that  $\psi_0(\cdot, \xi) \in C^0(Y)$  depends in a Lipschitz continuous way on  $\xi$ . We assumed  $f \in L^1(\mathbb{R}^n)$  and obtain therefore

$$\left| \hat{f}_0^\varepsilon(k) - F_0(k) \right| \leq C\varepsilon \|f\|_{L^1(\mathbb{R}^n)} \leq C\varepsilon,$$

uniformly in  $k \in K$ . Since  $K$  is compact, this provides also an  $L^1(K)$ -bound as in the statement of (2.15).

*Step 2:*  $k \in Z/\varepsilon \setminus K$ . We expand  $\psi_0(\cdot, \xi)$  in a Fourier series on the cube  $Y$ , treating  $\xi$  as a parameter,

$$\psi_0(y, \xi) = \sum_{l \in \mathbb{Z}^n} \alpha_l(\xi) e^{il \cdot y}.$$

Since we assumed that  $\psi_0(\cdot, \xi)$  is of class  $C^1$ , the Fourier series converges uniformly in  $y$ , for every parameter  $\xi$ . Since  $f$  is of class  $L^1(\mathbb{R}^n)$ , we may write

$$\begin{aligned} \hat{f}_0^\varepsilon(k) &= \int_{\mathbb{R}^n} f(x) e^{-ik \cdot x} \psi_0 \left( \frac{x}{\varepsilon}, \varepsilon k \right)^* dx \\ &= \lim_{L \rightarrow \infty} \sum_{l \in \mathbb{Z}^n, |l| \leq L} \alpha_l(\varepsilon k) \int_{\mathbb{R}^n} f(x) e^{-ik \cdot x} e^{il \cdot (x/\varepsilon)} dx. \end{aligned}$$

For each single term we find, since  $k - (l/\varepsilon) \notin K$  for  $\mathbb{Z}^n \ni l \neq 0$ ,

$$\int_{\mathbb{R}^n} f(x) e^{-ik \cdot x} e^{il \cdot (x/\varepsilon)} dx = (2\pi)^{n/2} F_0(k - (l/\varepsilon)) = \begin{cases} 0 & \text{for } l \neq 0, \\ (2\pi)^{n/2} F_0(k) & \text{for } l = 0. \end{cases}$$

Evaluation of the sum, we find

$$\hat{f}_0^\varepsilon(k) = \alpha_0(\varepsilon k) (2\pi)^{n/2} F_0(k) = 0.$$

In particular, we obtain the claim (2.16) about the support of  $\hat{f}_0^\varepsilon$ . This, in turn, implies also the  $L^1$ -estimate (2.15) for the difference on all of  $Z/\varepsilon$ .  $\square$

We want to use the theorem in order to simplify the representation of the solution  $u_0^\varepsilon$  of (2.14). We define a new approximation as

$$U^\varepsilon(x, t) := (2\pi)^{-n/2} \int_K F_0(k) e^{ik \cdot x} \operatorname{Re} \left( e^{it\sqrt{\mu_0^\varepsilon(k)}} \right) dk. \quad (2.18)$$

Theorem 2.2 allows to calculate, using once more (2.17) to compare  $w_0^\varepsilon(x, k) = \psi_0(x/\varepsilon, \varepsilon k) e^{ik \cdot x}$  with  $(2\pi)^{-n/2} e^{ik \cdot x}$ ,

$$\begin{aligned} \|u_0^\varepsilon - U^\varepsilon\|_{L^\infty((0, \infty) \times \mathbb{R}^n)} &= \left\| \int_K \hat{f}_0^\varepsilon(k) w_0^\varepsilon(x, k) \operatorname{Re} \left( e^{it\sqrt{\mu_0^\varepsilon(k)}} \right) dk - U^\varepsilon \right\|_{L^\infty((0, \infty) \times \mathbb{R}^n)} \\ &\leq \frac{1}{(2\pi)^{n/2}} \sup_{t \in (0, \infty)} \sup_{x \in \mathbb{R}^n} \left| \int_K \hat{f}_0^\varepsilon(k) e^{ik \cdot x} \operatorname{Re} \left( e^{it\sqrt{\mu_0^\varepsilon(k)}} \right) dk \right. \\ &\quad \left. - (2\pi)^{-n/2} \int_K F_0(k) e^{ik \cdot x} \operatorname{Re} \left( e^{it\sqrt{\mu_0^\varepsilon(k)}} \right) dk \right| + C\varepsilon \\ &\leq C \left\| \hat{f}_0^\varepsilon - F_0 \right\|_{L^1(Z/\varepsilon)} + C\varepsilon \leq C\varepsilon. \end{aligned}$$

We can combine this error estimate with the one obtained earlier for the difference  $\|u^\varepsilon - u_0^\varepsilon\|_{L^\infty((0, \infty), L^2(\mathbb{R}^n))}$ . We use, given two norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ , the new norm (weaker than both original norms)  $\|u\|_{X+Y} := \inf\{\|u_1\|_X + \|u_2\|_Y : u = u_1 + u_2\}$ . This allows to write the combined estimate as  $\|u^\varepsilon - U^\varepsilon\|_{L^\infty((0, \infty), (L^\infty + L^2)(\mathbb{R}^n))} \leq C\varepsilon$ .

### 2.3 Expansion of the dispersion relation

The next step is to replace the eigenvalue  $\mu_0$  by its Taylor series. We note that in a neighborhood of  $k = 0$  the eigenvalue  $\mu_0$  depends analytically on  $k$  and  $\mu_0(0) = \nabla \mu_0(0) = 0$ , cf. [7]. We denote the derivatives of  $\mu_0$  via  $A_{lm} = \frac{1}{2} \partial_{k_l} \partial_{k_m} \mu_0(0)$ ,  $B_{lmn} = \frac{1}{6} \partial_{k_l} \partial_{k_m} \partial_{k_n} \mu_0(0)$ , and  $C_{lmnq} = \frac{1}{24} \partial_{k_l} \partial_{k_m} \partial_{k_n} \partial_{k_q} \mu_0(0)$ . We will below study a symmetric situation in which  $B$  vanishes. We can then assume that the Taylor series of  $\mu_0$  in  $k$  around  $k = 0$  is given as

$$\mu_0(k) = \sum A_{lm} k_l k_m + \sum C_{lmnq} k_l k_m k_n k_q + O(|k|^5). \quad (2.19)$$

Here and below a bare sum is always over the repeated indices. The expansion corresponds to the following expansion of  $\mu_0^\varepsilon(k)$ ,

$$\mu_0^\varepsilon(k) = \frac{1}{\varepsilon^2} \mu_0(\varepsilon k) = \sum A_{lm} k_l k_m + \varepsilon^2 \sum C_{lmnq} k_l k_m k_n k_q + O(\varepsilon^3), \quad (2.20)$$

the error is of order  $\varepsilon^3$ , uniformly in  $k \in K$ .

In the spirit of this expansion, we next want to simplify further  $U^\varepsilon$  of (2.18). We define  $v^\varepsilon$  (compare page 992 of [23]) as

$$\begin{aligned} v^\varepsilon(x, t) &:= (2\pi)^{-n/2} \frac{1}{2} \sum_{\pm} \int_K F_0(k) e^{ik \cdot x} \exp \left( \pm it \sqrt{\sum A_{lm} k_l k_m} \right) \\ &\quad \times \exp \left( \pm \frac{i\varepsilon^2}{2} t \frac{\sum C_{lmnq} k_l k_m k_n k_q}{\sqrt{\sum A_{lm} k_l k_m}} \right) dk \end{aligned} \quad (2.21)$$

We arrive at the following approximation result. We repeat that the underlying observations are taken from [23].

**Corollary 2.1.** *Let  $f \in L^1(\mathbb{R}^n)$  have a Fourier-transform  $F_0$  supported on a compact set  $K \subset \mathbb{R}^n$ . Let  $u^\varepsilon$  be the solution of (1.1) and let  $v^\varepsilon$  be defined by (2.21). Then*

$$\sup_{t \in [0, T_0 \varepsilon^{-2}]} \|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^2(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)} \leq C\varepsilon. \quad (2.22)$$

*Proof.* The estimate for the difference  $u^\varepsilon - U^\varepsilon$  has been concluded after the definition of  $U^\varepsilon$  in (2.18). It remains to estimate the difference  $v^\varepsilon - U^\varepsilon$  in the same norm.

The Taylor expansion of the square root reads

$$\sqrt{a+c} = \sqrt{a} + \frac{1}{2\sqrt{a}}c + O(|c|^2).$$

This implies that the definitions of  $U^\varepsilon(t)$  and  $v^\varepsilon(t)$  coincide, except for a factor of the form

$$\exp(\pm itO(\varepsilon^4)) = 1 + O(\varepsilon^2),$$

uniformly in  $t$  for  $t \in [0, T_0 \varepsilon^{-2}]$ . Because of  $F_0 \in L^\infty(\mathbb{R}^n)$  and boundedness of  $K$ , this implies (2.22).  $\square$

In view of Corollary 2.1, it will no longer be necessary to work with  $u^\varepsilon$ , the solution to the original wave equation in a heterogeneous medium. We will, instead, restrict ourselves to the analysis of the function  $v^\varepsilon$  defined by (2.21).

Note that Taylor expansions of Bloch eigenvalues are commonly used also in the derivation of effective equations for envelopes of nonlinear waves in periodic structures, see e.g. [8, 9].

## 2.4 Symmetries

The structure of the three tensors  $A$ ,  $B$  and  $C$ , defined via the expansion of  $\mu_0(k)$ , is very simple if we consider only symmetric material functions  $a_Y$ . Indeed, we will see that  $A$ ,  $B$ , and  $C$  are fully characterized by three real numbers  $a^*$ ,  $\alpha$ , and  $\beta$ .

We assume that  $a_Y$  is symmetric with respect to reflections across a hyperplane  $\{y_j = 0\}$ ,  $j \in \{0, \dots, n\}$ , and invariant under coordinate permutations. To be more precise, we introduce the following transformation of  $\mathbb{R}^n$ , defined for  $y = (y_1, \dots, y_n)$  as

$$\begin{aligned} S_i(y) &= (y_1, \dots, y_{i-1}, -y_i, y_{i+1}, \dots, y_n), \\ R_{ij}(y) &= (y_1, \dots, y_{i-1}, y_j, y_{i+1}, \dots, y_{j-1}, y_i, y_{j+1}, \dots, y_n). \end{aligned}$$

Our symmetry assumption on  $a_Y$  can now be formulated as

$$a_Y(y) = a_Y(S_i(y)) = a_Y(R_{ij}(y)) \quad \text{for all } i, j \in \{1, \dots, n\} \text{ and all } y \in \mathbb{R}^n. \quad (2.23)$$

As we show next, the symmetry properties of  $a_Y$  in  $y$  imply the identical symmetry properties of  $\mu_0$  in  $k$ ,

$$\mu_0(k) = \mu_0(S_i(k)) = \mu_0(R_{ij}(k)) \quad \text{for all } i, j \in \{1, \dots, n\} \text{ and all } k \in K. \quad (2.24)$$

In fact, (2.24) holds also for all functions  $\mu_m$ , but we exploit here only the symmetry of  $\mu_0$ . To show (2.24), we express  $\mu_0(k)$  with the variational characterization, see Theorem XIII.2 in [21], as

$$\mu_0(k) = \min_{\substack{w \in H_{\text{per}}^1(Y) \\ \|w\|_{L^2(Y)}=1}} I(w, k), \quad \text{where } I(w, k) := \int_Y a_Y(y) |(\nabla + ik)w|^2 dy. \quad (2.25)$$

Using the symmetry of  $a_Y$ , we can calculate

$$\begin{aligned} I(w, S_i(k)) &= \int_Y a_Y(y) |[(\nabla + iS_i(k))w](y)|^2 dy \\ &= \int_{S_i^{-1}(Y)} a_Y(\tilde{y}) |[(\nabla + iS_i(k))w](S_i(\tilde{y}))|^2 d\tilde{y} \\ &= \int_Y a_Y(\tilde{y}) |S_i([(\nabla + ik)(w \circ S_i)](\tilde{y}))|^2 d\tilde{y} = I(w \circ S_i, k). \end{aligned} \quad (2.26)$$

Minimizing over the functions  $w \circ S_i$  provides the same result as minimizing over  $w$ , since with  $w \in H_{\text{per}}^1(Y)$  also  $w \circ S_i \in H_{\text{per}}^1(Y)$ . This provides (2.24) for  $S_i$ . The calculation for  $R_{ij}$  is identical.

As a consequence of the symmetry, we obtain the following characterization of the Taylor expansion coefficients.

**Lemma 2.2.** *Let  $a_Y$  satisfy the symmetries (2.23). Then the tensors  $A$ ,  $B$  and  $C$ , defined in (2.19), satisfy*

$$\begin{aligned} A_{ii} &= A_{11} =: a^*, & A_{ij} &= 0, & B &= 0, \\ C_{iii} &= C_{111} =: \alpha, & C_{ijij} &= C_{ijji} = C_{iijj} = C_{1122} =: \beta \end{aligned}$$

for all  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ . All entries of  $C$ , that are not mentioned above, vanish.

*Proof.* The proof uses symmetry (2.24). The symmetry under  $S_i$  implies that  $\mu_0$  is an even function. Thus all derivatives of  $\mu_0$  with an odd number of derivatives in one variable vanish at  $k = 0$ . This proves  $A_{ij} = 0$ ,  $B = 0$ , and, e.g.,  $C_{iiij} = 0$ . The fact that derivatives can be interchanged provides, e.g.,  $C_{iijj} = C_{ijij}$ .

The symmetry under  $R_{ij}$  allows to calculate

$$\partial_{k_i}^2 \mu_0(k) = \partial_{k_i}^2 (\mu_0 \circ R_{ij})(k) = [\partial_{k_j}^2 \mu_0](R_{ij}(k)).$$

Evaluating in  $k = 0$  provides  $A_{ii} = A_{jj}$ . The analogous calculation for fourth derivatives shows, e.g.,  $C_{iii} = C_{jjj}$ . This proves the claim in the two-dimensional case.

For  $n \geq 3$  we can calculate again with the symmetry under  $R_{ij}$

$$\partial_{k_i}^2 \partial_{k_j}^2 \mu_0(k) = \partial_{k_i}^2 \partial_{k_j}^2 (\mu_0 \circ R_{jl})(k) = [\partial_{k_i}^2 \partial_{k_l}^2 \mu_0](R_{jl}(k)).$$

Evaluating at  $k = 0$ , we obtain  $C_{iijj} = C_{iill}$  for all indices  $i, j, l \leq n$ .  $\square$

### 3 A well-posed weakly dispersive equation

A weakly dispersive equation that is related to the definition of  $v^\varepsilon$  is (at this point, we correct a typo of [23] regarding the sign before  $C$ )

$$\partial_t^2 u = AD^2 u - \varepsilon^2 CD^4 u. \quad (3.1)$$

Indeed, when applied to  $v^\varepsilon$ , the operator  $AD^2$  produces the factor  $-A_{lm}k_l k_m$  under the integral, and the operator  $-\varepsilon^2 CD^4$  produces the factor  $-\varepsilon^2 C_{lmnq}k_l k_m k_n k_q$ . The second time derivative produces the factor

$$-A_{lm}k_l k_m - \varepsilon^2 C_{lmnq}k_l k_m k_n k_q - (\varepsilon^4/4)(C_{lmnq}k_l k_m k_n k_q)^2/(A_{lm}k_l k_m)$$

under the integral. Therefore, up to an error of order  $\varepsilon^4$ , the function  $v^\varepsilon$  solves (3.1).

We emphasize that, in general, (3.1) cannot be used as an effective dispersive model. The fourth order operator  $-CD^4$  on the right hand side can be positive such that (3.1) is ill-posed. In the one-dimensional setting,  $C < 0$  is shown in [18], hence the equation is necessarily ill-posed. Section 4.2 includes a two-dimensional numerical example where the numbers  $\alpha$  and  $\beta$ , describing  $C$ , satisfy  $\alpha < 0$  and  $\beta > 0$ . Moreover, there holds  $3\beta < |\alpha|$ , such that  $-CD^4$  is a positive operator.

As a consequence, even though  $v^\varepsilon$  solves (3.1) up to an error of order  $\varepsilon^4$ , we cannot conclude that solutions to this equation provide approximations of  $v^\varepsilon$ . Even worse, it may be impossible to construct any solution of (3.1).

#### 3.1 Decomposition of the operator for symmetric media

As indicated in the introduction, our aim is now to replace (3.1) by a well-posed equation, which is equivalent in all relevant powers of  $\varepsilon$ . We therefore start from the two tensors  $A = a^* \text{id} \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{n \times n \times n \times n}$  of Lemma 2.2 and consider the operator

$$CD^4 = \sum_{ijkl} C_{ijkl} \partial_i \partial_j \partial_k \partial_l = \alpha \sum_{i=1}^n \partial_i^4 + 3\beta \sum_{\substack{i,j=1 \\ i \neq j}}^n \partial_i^2 \partial_j^2. \quad (3.2)$$

To avoid confusion, we note that  $\sum_{i \neq j} = 2 \sum_{i < j}$ . Our aim is to construct coefficients  $E \in \mathbb{R}^{n \times n}$  and  $F \in \mathbb{R}^{n \times n \times n \times n}$  such that the differential operator can be re-written as

$$-CD^4 = ED^2 AD^2 - FD^4, \quad (3.3)$$

where  $E$  and  $F$  are positive semidefinite and symmetric, i.e.

$$\sum_{i,j,k,l=1}^n F_{ijkl} \xi_{ij} \xi_{kl} \geq 0 \quad \text{for every } \xi \in \mathbb{R}^{n \times n} \quad \text{and} \quad F_{ijkl} = F_{klij} \quad (3.4)$$

and  $\sum_{i,j=1}^n E_{ij} \eta_i \eta_j \geq 0$  for every  $\eta \in \mathbb{R}^n$  and  $E_{ij} = E_{ji}$  for  $i, j, k, l \in \{1, \dots, n\}$ . The decomposition result (3.3) allows, using the lowest order of (3.1), to re-write the operator in the evolution equation formally as

$$-\varepsilon^2 CD^4 u = \varepsilon^2 ED^2 AD^2 u - \varepsilon^2 FD^4 u = \varepsilon^2 ED^2 \partial_t^2 u - \varepsilon^2 FD^4 u + O(\varepsilon^4). \quad (3.5)$$

With this replacement in equation (3.1), we obtain the well-posed equation (1.5).

**Lemma 3.1** (Decomposability). *Let  $A \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{n \times n \times n \times n}$  be as in Lemma 2.2, given by three constants  $a^* > 0$ ,  $\alpha, \beta \in \mathbb{R}$ , in particular with  $CD^4$  given by (3.2). Then there exist symmetric and positive semidefinite tensors  $E \in \mathbb{R}^{n \times n}$  and  $F \in \mathbb{R}^{n \times n \times n \times n}$  such that  $CD^4$  can be written as in (3.3).*

Using  $\{a\}_+ := \max\{a, 0\}$  to denote the positive part of a number  $a$ , a possible choice of  $E$  and  $F$  is

$$E_{ii} = \frac{1}{a^*} (\{-\alpha\}_+ + 3\{-\beta\}_+), \quad E_{ij} = 0, \quad (3.6)$$

$$F_{iiii} = \{\alpha\}_+ + 3\{-\beta\}_+, \quad F_{ijij} = \{-\alpha\}_+ + 3\{\beta\}_+, \quad (3.7)$$

for all  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ . All other entries of  $F$  are set to zero.

With (3.6)–(3.7), we introduce the two differential operators

$$ED^2 = \frac{1}{a^*} (\{-\alpha\}_+ + 3\{-\beta\}_+) \sum_{i=1}^n \partial_i^2 = \frac{1}{a^*} (\{-\alpha\}_+ + 3\{-\beta\}_+) \Delta,$$

$$FD^4 = (\{\alpha\}_+ + 3\{-\beta\}_+) \sum_{i=1}^n \partial_i^4 + (\{-\alpha\}_+ + 3\{\beta\}_+) \sum_{i,j=1, i \neq j}^n \partial_i^2 \partial_j^2.$$

Since  $\alpha$  and  $\beta$  are real numbers, there are four different possibilities for the signs of  $\alpha$  and  $\beta$ . Distinguishing these four cases, we can write the two differential operators in very simple expressions.

**Remark 3.1.** *The operators  $ED^2$  and  $FD^4$  of (3.6)–(3.7) are given as follows.*

**Case 1.**  $\alpha \leq 0, \beta \leq 0$ :

$$ED^2 = \frac{1}{a^*} (|\alpha| + 3|\beta|) \Delta \quad \text{and} \quad FD^4 = 3|\beta| \sum_{i=1}^n \partial_i^4 + |\alpha| \sum_{i,j=1, i \neq j}^n \partial_i^2 \partial_j^2$$

**Case 2.**  $\alpha \leq 0, \beta > 0$ :

$$ED^2 = \frac{|\alpha|}{a^*} \Delta \quad \text{and} \quad FD^4 = (|\alpha| + 3\beta) \sum_{i,j=1, i \neq j}^n \partial_i^2 \partial_j^2.$$

**Case 3.**  $\alpha > 0, \beta \leq 0$ :

$$ED^2 = \frac{3|\beta|}{a^*} \Delta \quad \text{and} \quad FD^4 = (\alpha + 3|\beta|) \sum_{i=1}^n \partial_i^4$$

**Case 4.**  $\alpha \geq 0, \beta \geq 0$ :

$$ED^2 = 0 \quad \text{and} \quad FD^4 = \alpha \sum_{i=1}^n \partial_i^4 + 3\beta \sum_{i,j=1, i \neq j}^n \partial_i^2 \partial_j^2 = CD^4.$$

We note that the first two cases (with  $\alpha \leq 0$ ) are the relevant ones in our numerical examples.

*Proof of Lemma 3.1. Step 1. Properties of  $E$  and  $F$ .* By definition,  $E$  is a nonnegative multiple of the identity in  $\mathbb{R}^n$ . The tensor is therefore positive semidefinite and symmetric. Also  $F$  is symmetric by definition. For  $\xi \in \mathbb{R}^{n \times n}$  holds

$$\begin{aligned} & \sum_{i,j,k,l=1}^n F_{ijkl} \xi_{ij} \xi_{kl} \\ &= \sum_{i=1}^n (\{\alpha\}_+ + 3\{-\beta\}_+) (\xi_{ii})^2 + \sum_{i,j=1, i \neq j}^n (\{-\alpha\}_+ + 3\{\beta\}_+) (\xi_{ij})^2 \geq 0. \end{aligned}$$

Hence  $F$  is also positive semidefinite.

*Step 2. Decomposition property.* It remains to show  $-CD^4 = ED^2AD^2 - FD^4$ . For that purpose we calculate the right hand side

$$\begin{aligned} & ED^2AD^2 - FD^4 \\ &= \frac{1}{a^*} (\{-\alpha\}_+ + 3\{-\beta\}_+) \sum_{i=1}^n \partial_i^2 \left( \sum_{j=1}^n a^* \partial_j^2 \right) - (\{\alpha\}_+ + 3\{-\beta\}_+) \sum_{i=1}^n \partial_i^4 \\ &\quad - (\{-\alpha\}_+ + 3\{\beta\}_+) \sum_{i,j=1, i \neq j}^n \partial_i^2 \partial_j^2 \\ &= (\{-\alpha\}_+ + 3\{-\beta\}_+) \sum_{i=1}^n \partial_i^4 + (\{-\alpha\}_+ + 3\{-\beta\}_+) \sum_{i,j=1, i \neq j}^n \partial_i^2 \partial_j^2 \\ &\quad - (\{\alpha\}_+ + 3\{-\beta\}_+) \sum_{i=1}^n \partial_i^4 - (\{-\alpha\}_+ + 3\{\beta\}_+) \sum_{i,j=1, i \neq j}^n \partial_i^2 \partial_j^2 \\ &= -\alpha \sum_{i=1}^n \partial_i^4 - 3\beta \sum_{i,j=1, i \neq j}^n \partial_i^2 \partial_j^2 = -CD^4. \end{aligned}$$

This is the desired decomposition (3.3).  $\square$

### 3.2 An approximation result

With the subsequent theorem, we provide the central error estimate for our main result. We start from two tensors  $A$  and  $C$  (in the application of the theorem they are defined by (2.19)), and assume that  $C$  is decomposable with tensors  $E$  and  $F$ . With these four tensors we can study two objects: The solution  $w^\varepsilon$  of (1.5), and the function  $v^\varepsilon$ , defined by the representation formula (2.21). Our next theorem compares these two objects.

**Theorem 3.1.** *Let  $A, C, E, F$  be tensors with the properties:  $A$  is symmetric and positive definite,  $\sum_{ij} A_{ij} \xi_i \xi_j \geq \gamma |\xi|^2$  for some  $\gamma > 0$ ,  $E$  and  $F$  are positive semidefinite and symmetric,  $C$  allows the decomposition (3.3). Then the following holds.*



1. **Well-posedness.** Let  $R \in L^1(0, T_0\varepsilon^{-2}; L^2(\mathbb{R}^n))$  be a right hand side and let  $f \in H^2(\mathbb{R}^n)$  be an initial datum. We study an inhomogeneous version of equation (1.5),

$$\begin{aligned} \partial_t^2 w^\varepsilon(x, t) - AD^2 w^\varepsilon(x, t) - \varepsilon^2 \partial_t^2 ED^2 w^\varepsilon(x, t) + \varepsilon^2 FD^4 w^\varepsilon(x, t) &= R(x, t), \\ w^\varepsilon(x, 0) &= f(x), \quad \partial_t w^\varepsilon(x, 0) = 0. \end{aligned} \quad (3.8)$$

for  $x \in \mathbb{R}^n$  and  $t \in (0, T_0\varepsilon^{-2})$ . This equation has a unique solution  $w^\varepsilon \in L^\infty(0, T_0\varepsilon^{-2}; H^2(\mathbb{R}^n))$ .

2. **Approximation.** Let  $v^\varepsilon$  be defined by (2.21) with  $F_0$  and  $f$  related by (1.3). Let  $w^\varepsilon$  be a solution of (3.8) to  $R \equiv 0$ . Then

$$\sup_{t \in [0, T_0\varepsilon^{-2}]} \|\partial_t(v^\varepsilon - w^\varepsilon)(\cdot, t)\|_{L^2(\mathbb{R}^n)} + \sup_{t \in [0, T_0\varepsilon^{-2}]} \|\nabla(v^\varepsilon - w^\varepsilon)(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon^2, \quad (3.9)$$

where  $C > 0$  denotes a constant that depends on  $f$  and the coefficients, but is independent of  $\varepsilon$ .

*Proof.* *Well-posedness of problem (3.8).* We use the following concept of weak solutions. We say that  $w^\varepsilon \in L^\infty(0, T_0\varepsilon^{-2}; H^2(\mathbb{R}^n))$  with the property  $\partial_t w^\varepsilon \in L^\infty(0, T_0\varepsilon^{-2}; H^1(\mathbb{R}^n))$  is a weak solution, if it satisfies  $w^\varepsilon(x, 0) = f(x)$  in the sense of traces and if

$$\begin{aligned} \int_0^{T_0\varepsilon^{-2}} \int_{\mathbb{R}^n} R \phi &= \int_0^{T_0\varepsilon^{-2}} \int_{\mathbb{R}^n} \{-\partial_t w^\varepsilon \partial_t \phi + \nabla \phi \cdot A \nabla w^\varepsilon\} \\ &\quad + \varepsilon^2 \int_0^{T_0\varepsilon^{-2}} \int_{\mathbb{R}^n} \{-\nabla(\partial_t \phi) \cdot E \nabla(\partial_t w^\varepsilon) + D^2 \phi : F D^2 w^\varepsilon\} \end{aligned} \quad (3.10)$$

for every test-function  $\phi \in C_c^1([0, T_0\varepsilon^{-2}]; H^2(\mathbb{R}^n))$ . Here  $D^2 \phi : F D^2 w^\varepsilon$  denotes the tensor product of  $D^2 \phi$  and  $F D^2 w^\varepsilon$ ,

$$D^2 \phi : F D^2 w^\varepsilon := \sum_{i,j,k,l=1}^n \partial_i \partial_j \phi F_{ijkl} \partial_k \partial_l w^\varepsilon.$$

We prove the existence of a weak solution to problem (3.8) with a Galerkin scheme. We use a countable basis  $\{\psi^k\}_{k \in \mathbb{N}}$  of the separable space  $H^1(\mathbb{R}^n)$  and the finite-dimensional sub-spaces  $V_K := \text{span}\{\psi^1, \dots, \psi^K\} \subset H^1(\mathbb{R}^n)$ . The basis  $\{\psi^k\}_{k \in \mathbb{N}}$  is chosen in such a way that the functions  $\psi^k$  are of class  $H^2(\mathbb{R}^n)$  and such that the family of  $L^2$ -orthogonal projections  $P_K$  onto  $V_K$  are bounded as maps  $P_K : H^2(\mathbb{R}^n) \rightarrow H^2(\mathbb{R}^n)$ . For every  $K \in \mathbb{N}$  we search for approximative solutions  $w_K^\varepsilon$  of the form

$$w_K^\varepsilon : [0, T_0\varepsilon^{-2}] \rightarrow V_K, \quad w_K^\varepsilon(t) = \sum_{k=1}^K a_k^\varepsilon(t) \psi^k$$

with coefficients  $a_k^\varepsilon : [0, T_0\varepsilon^{-2}] \rightarrow \mathbb{R}$ . We demand that  $w_K^\varepsilon$  solves (3.8) in the weak sense, however, only for test-functions in the  $K$ -dimensional space  $V_K$ ,

$$\begin{aligned} \int_{\mathbb{R}^n} R\psi^k &= \int_{\mathbb{R}^n} \{\partial_t^2 w_K^\varepsilon \psi^k + \nabla \psi^k \cdot A \nabla w_K^\varepsilon\} \\ &\quad + \varepsilon^2 \int_{\mathbb{R}^n} \{\nabla \psi^k \cdot E \nabla (\partial_t^2 w_K^\varepsilon) + D^2 \psi^k : F D^2 w_K^\varepsilon\} \end{aligned} \quad (3.11)$$

for every  $k \in \{1, \dots, K\}$ . For the initial data we demand that  $\langle w_K^\varepsilon|_{t=0}, \psi^k \rangle_{L^2(\mathbb{R}^n)} = \langle f, \psi^k \rangle_{L^2(\mathbb{R}^n)}$  and  $\langle \partial_t w_K^\varepsilon|_{t=0}, \psi^k \rangle_{L^2(\mathbb{R}^n)} = 0$ . For every  $K \in \mathbb{N}$ , equation (3.11) is a  $K$ -dimensional system of ordinary differential equations of second order for the coefficient vector  $(a_1^\varepsilon(t), \dots, a_K^\varepsilon(t))$ , which can be solved uniquely. This provides the approximative solutions  $w_K^\varepsilon$ .

We now derive  $K$ -independent a priori estimates for the sequence  $w_K^\varepsilon$ . For that purpose we test equation (3.8) with  $\partial_t w_K^\varepsilon$  (more precisely, we multiply (3.11) with  $\partial_t a_k^\varepsilon$  and take the sum over  $k$ ). Exploiting the symmetry of  $A, E$  and  $F$  we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} R \partial_t w_K^\varepsilon &= \frac{1}{2} \partial_t \int_{\mathbb{R}^n} \{|\partial_t w_K^\varepsilon|^2 + \nabla w_K^\varepsilon \cdot A \nabla w_K^\varepsilon\} \\ &\quad + \varepsilon^2 \frac{1}{2} \partial_t \int_{\mathbb{R}^n} \{\nabla(\partial_t w_K^\varepsilon) \cdot E \nabla(\partial_t w_K^\varepsilon) + D^2 w_K^\varepsilon : F D^2 w_K^\varepsilon\}. \end{aligned} \quad (3.12)$$

We next integrate (3.12) over  $[0, t_0]$ , where  $t_0 \in [0, T_0\varepsilon^{-2}]$  is arbitrary. We exploit the initial condition  $w_K^\varepsilon|_{t=0} = f_K$ , where  $f_K$  is the  $L^2$ -projection of  $f$  onto  $V_K$ . The other initial condition is  $\partial_t w_K^\varepsilon|_{t=0} = 0$  and we arrive at

$$\begin{aligned} 2 \int_0^{t_0} \int_{\mathbb{R}^n} R \partial_t w_K^\varepsilon &+ \int_{\mathbb{R}^n} \nabla f_K \cdot A \nabla f_K + \varepsilon^2 \int_{\mathbb{R}^n} D^2 f_K : F D^2 f_K \\ &= \int_{\mathbb{R}^n} \{|\partial_t w_K^\varepsilon|_{t=t_0}|^2 + \nabla w_K^\varepsilon|_{t=t_0} \cdot A \nabla w_K^\varepsilon|_{t=t_0}\} \\ &\quad + \varepsilon^2 \int_{\mathbb{R}^n} \{\nabla(\partial_t w_K^\varepsilon)|_{t=t_0} \cdot E \nabla(\partial_t w_K^\varepsilon)|_{t=t_0} + D^2 w_K^\varepsilon|_{t=t_0} : F D^2 w_K^\varepsilon|_{t=t_0}\} \\ &\geq \|\partial_t w_K^\varepsilon(\cdot, t_0)\|_{L^2(\mathbb{R}^n)}^2 + \gamma \|\nabla w_K^\varepsilon(\cdot, t_0)\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (3.13)$$

In the last line we exploited that  $A$  is positive definite with parameter  $\gamma > 0$  and that  $E$  and  $F$  are positive semi-definite. Introducing  $Y(t) := \|\partial_t w_K^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2 + \gamma \|\nabla w_K^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2$  for the right hand side of (3.13) and  $Y_0 := \int_{\mathbb{R}^n} \{\nabla f_K \cdot A \nabla f_K + \varepsilon^2 D^2 f_K : F D^2 f_K\}$ , we can calculate with the Cauchy-Schwarz inequality

$$\begin{aligned} Y(t) &\leq 2 \int_0^t \|R(\cdot, s)\|_{L^2(\mathbb{R}^n)} \|\partial_t w_K^\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}^n)} ds + Y_0 \\ &\leq 2 \int_0^t \|R(\cdot, s)\|_{L^2(\mathbb{R}^n)} \sqrt{Y(s)} ds + Y_0. \end{aligned} \quad (3.14)$$

We claim that a Gronwall-type argument leads from inequality (3.14) to the estimate

$$Y(t) \leq 2Y_0 + 2 \left( \int_0^t \|R(\cdot, s)\|_{L^2(\mathbb{R}^n)} ds \right)^2. \quad (3.15)$$

For the sake of clarity we postpone the justification of this implication to the end of the proof. With inequality (3.15) at hand we finally obtain the following a priori estimate

$$\begin{aligned} \sup_{t \in [0, T_0 \varepsilon^{-2}]} Y(t) &= \sup_{t \in [0, T_0 \varepsilon^{-2}]} \left\{ \|\partial_t w_K^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2 + \gamma \|\nabla w_K^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2 \right\} \\ &\leq 2Y_0 + 2\|R\|_{L^1(0, T_0 \varepsilon^{-2}; L^2(\mathbb{R}^n))}^2 \\ &\leq 2(C(A) + \varepsilon^2 C(F))\|f\|_{H^2(\mathbb{R}^n)}^2 + 2\|R\|_{L^1(0, T_0 \varepsilon^{-2}; L^2(\mathbb{R}^n))}^2. \end{aligned} \quad (3.16)$$

The bound in (3.16) is independent of  $K$ . Hence, possibly after passing to a subsequence, we may consider the weak limit  $K \rightarrow \infty$  of solutions  $w_K^\varepsilon$  of the Galerkin scheme. Due to the linearity of the problem, the limit provides a solution  $w^\varepsilon \in L^\infty(0, T_0 \varepsilon^{-2}; H^1(\mathbb{R}^n))$  with  $\partial_t w^\varepsilon \in L^\infty(0, T_0 \varepsilon^{-2}; L^2(\mathbb{R}^n))$  to (3.8) in the sense of distributions. Furthermore,  $w^\varepsilon$  satisfies exactly the same a priori estimates as its approximations  $w_K^\varepsilon$ . By differentiating (3.8) with respect to  $x$ , one discovers that  $w^\varepsilon$  has in fact higher spatial regularity and that the distributional solution  $w^\varepsilon$  is in fact a weak solution in the sense of (3.10). Note that the uniqueness of solutions to problem (3.8) is a direct consequence of the a priori estimate (3.16). Hence, the weakly dispersive problem is well-posed.

*Proof of the approximation result (3.9).* By applying the differential operator  $\partial_t^2 - AD^2 - \varepsilon^2 \partial_t^2 ED^2 + \varepsilon^2 FD^4$  to  $v^\varepsilon$ , which is explicitly given in (2.21), one immediately discovers that  $v^\varepsilon$  solves Equation (3.8) with a right hand side of order  $\varepsilon^4$ . More precisely, we calculate first with the decomposition of the operator  $-CD^4 = ED^2 AD^2 - FD^4$

$$\partial_t^2 v^\varepsilon - AD^2 v^\varepsilon = -\varepsilon^2 CD^4 v^\varepsilon + \varepsilon^4 \tilde{R}^\varepsilon = \varepsilon^2 ED^2 AD^2 v^\varepsilon - \varepsilon^2 FD^4 v^\varepsilon + \varepsilon^4 \tilde{R}^\varepsilon,$$

where the error term comes from the double differentiation of the last factor of  $v^\varepsilon$  with respect to time,

$$\begin{aligned} \tilde{R}^\varepsilon &:= -\frac{1}{8}(2\pi)^{-n/2} \sum_{\pm} \int_{k \in K} \frac{(\sum C_{lmnq} k_l k_m k_n k_q)^2}{\sum A_{lm} k_l k_m} F_0(k) \\ &\times \exp\left(ik \cdot x \pm i\sqrt{\sum A_{lm} k_l k_m} t\right) \exp\left(\pm \frac{i\varepsilon^2}{2} t \frac{\sum C_{lmnq} k_l k_m k_n k_q}{\sqrt{\sum A_{lm} k_l k_m}}\right) dk. \end{aligned}$$

With this preparation we can now evaluate the application of the full differential operator as

$$\begin{aligned} \partial_t^2 v^\varepsilon - AD^2 v^\varepsilon + \varepsilon^2 FD^4 v^\varepsilon - \varepsilon^2 \partial_t^2 ED^2 v^\varepsilon \\ = \varepsilon^2 ED^2 AD^2 v^\varepsilon + \varepsilon^4 \tilde{R}^\varepsilon - \varepsilon^2 \partial_t^2 ED^2 v^\varepsilon \\ = \varepsilon^2 ED^2 (AD^2 v^\varepsilon - \partial_t^2 v^\varepsilon) + \varepsilon^4 \tilde{R}^\varepsilon \\ = \varepsilon^4 ED^2 (CD^4 v^\varepsilon - \varepsilon^2 \tilde{R}^\varepsilon) + \varepsilon^4 \tilde{R}^\varepsilon =: R^\varepsilon. \end{aligned} \quad (3.17)$$

In particular,  $\sup_{t \in [0, T_0 \varepsilon^{-2}]} \|R^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq \tilde{C} \varepsilon^4$  for some  $\varepsilon$ -independent constant  $\tilde{C}$ . Due to the linearity of the problem and the fact that  $w^\varepsilon$  is a solution to (3.8) with  $R \equiv 0$ , the difference  $v^\varepsilon - w^\varepsilon$  solves equation (3.17)

$$\partial_t^2 (v^\varepsilon - w^\varepsilon) - AD^2 (v^\varepsilon - w^\varepsilon) + \varepsilon^2 FD^4 (v^\varepsilon - w^\varepsilon) - \varepsilon^2 \partial_t^2 ED^2 (v^\varepsilon - w^\varepsilon) = R^\varepsilon,$$

with vanishing initial data  $(v^\varepsilon - w^\varepsilon)(\cdot, 0) = \partial_t(v^\varepsilon - w^\varepsilon)(\cdot, 0) = 0$ .

By applying the a priori estimate (3.16) to the difference  $(v^\varepsilon - w^\varepsilon)$  we obtain

$$\begin{aligned} & \sup_{t \in [0, T_0 \varepsilon^{-2}]} \left\{ \|\partial_t(v^\varepsilon - w^\varepsilon)(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2 + \gamma \|\nabla(v^\varepsilon - w^\varepsilon)(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2 \right\} \\ & \leq 2 \|R^\varepsilon\|_{L^1(0, T_0 \varepsilon^{-2}; L^2(\mathbb{R}^n))}^2 \leq 2(T_0 \varepsilon^{-2} \|R^\varepsilon\|_{L^\infty(0, T_0 \varepsilon^{-2}; L^2(\mathbb{R}^n))})^2 \leq C \varepsilon^4, \end{aligned}$$

where in the last step we exploited that  $\|R^\varepsilon\|_{L^\infty(0, T_0 \varepsilon^{-2}; L^2(\mathbb{R}^n))} \leq C \varepsilon^4$ . This implies (3.9).

*Proof of the Gronwall-type Inequality (3.15).* Let  $Y : [0, T] \rightarrow [0, \infty)$  be a function such that, for a constant  $Y_0 \geq 0$ , the relation

$$Y(t) \leq 2 \int_0^t \|R(\cdot, s)\|_{L^2(\mathbb{R}^n)} \sqrt{Y(s)} ds + Y_0 \quad (3.18)$$

holds for all times  $t \in [0, T]$ . We claim that then

$$Y(t) \leq 2 \left( \int_0^t \|R(\cdot, s)\|_{L^2(\mathbb{R}^n)} ds \right)^2 + 2Y_0 \quad (3.19)$$

holds for all times  $t \in [0, T]$ .

For the proof we define  $Z(t)$  to be the integral on the right hand side of (3.18),

$$Z(t) := 2 \int_0^t \|R(\cdot, s)\|_{L^2(\mathbb{R}^n)} \sqrt{Y(s)} ds.$$

Then  $Z(0) = 0$  and, due to the assumption (3.18),

$$\frac{d}{dt} Z(t) = 2 \|R(\cdot, t)\|_{L^2(\mathbb{R}^n)} \sqrt{Y(t)} \leq 2 \|R(\cdot, t)\|_{L^2(\mathbb{R}^n)} \sqrt{Z(t) + Y_0}.$$

We conclude that

$$\frac{d}{dt} \left( \sqrt{Z(t) + Y_0} \right) = \left( 2 \sqrt{Z(t) + Y_0} \right)^{-1} \frac{d}{dt} Z(t) \leq \|R(\cdot, t)\|_{L^2(\mathbb{R}^n)}.$$

Integrating this relation over  $[0, t]$  we obtain, recalling  $Z(0) = 0$ ,

$$\sqrt{Z(t) + Y_0} - \sqrt{Y_0} \leq \int_0^t \|R(\cdot, s)\|_{L^2(\mathbb{R}^n)} ds.$$

By evaluating the square we find

$$\begin{aligned} Z(t) + Y_0 & \leq \left( \sqrt{Y_0} + \int_0^t \|R(\cdot, s)\|_{L^2(\mathbb{R}^n)} ds \right)^2 \\ & \leq 2Y_0 + 2 \left( \int_0^t \|R(\cdot, s)\|_{L^2(\mathbb{R}^n)} ds \right)^2, \end{aligned}$$

and therefore the claimed result (3.19), since  $Y(t) \leq Z(t) + Y_0$  holds by assumption.  $\square$

**The main theorem.** Theorem 1.1 is a consequence of the above results.

*Proof.* We have seen in Lemma 2.1, that the solution  $u^\varepsilon$  permits the expansion (2.10) in Bloch-waves. The assumptions  $a_Y \in C^1(Y)$  and  $f \in H^1(\mathbb{R}^n)$  as in (1.3) are satisfied in the situation of Theorem 1.1.

In Theorem 2.1 we have seen that only the term  $m = 0$  in the sum must be considered. Again, the assumptions are satisfied,  $a_Y \in C^1(Y)$  and  $H^1(\mathbb{R}^n)$ -convergence of the Bloch-series of  $f$ .

The assumptions of Theorem 1.1 imply  $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$  and  $F_0 \in L^\infty(\mathbb{R}^n)$ , hence Theorem 2.2 can be applied. Furthermore, we assumed that  $\psi_0(\cdot, \xi) \in C^1(Y, \mathbb{R})$  is bounded independent of  $\xi \in Z$ . This is satisfied by elliptic regularity theory: Hölder-continuous coefficients yield bounds for solutions of divergence-form operators in  $C^{1,\alpha}$ , see Giaquinta, [16] Chapter III, Theorem 3.2. We recall that  $Z = (-1/2, 1/2)^n$  is bounded.

We concluded with (2.22) a smallness result,  $\|u^\varepsilon - v^\varepsilon\|$  is of order  $\varepsilon^2$ . The norms coincide with the ones in the claimed result (1.6) for  $\|u^\varepsilon - w^\varepsilon\|$ , where we only claim the order  $\varepsilon$  for the error.

Finally, Theorem 3.1 provides the well-posedness claim and the smallness (3.9) of the error  $\|v^\varepsilon - w^\varepsilon\|$  of order  $\varepsilon^2$ . We note that (3.9) controls only the norms of derivatives, but the subsequent Lemma 3.2 provides the estimate for  $\|v^\varepsilon - w^\varepsilon\|$  of order  $\varepsilon$  in the desired norm of (1.6).  $\square$

**Lemma 3.2.** *For  $n \geq 1$  and  $T > 0$  fixed, let  $g^\varepsilon : \mathbb{R}^n \times [0, T/\varepsilon^2] \rightarrow \mathbb{R}$  be a sequence of functions with  $g^\varepsilon(\cdot, 0) \equiv 0$ . Then, with a constant  $C > 0$ , there holds an  $\varepsilon$ -independent estimate*

$$\begin{aligned} & \sup_{t \in [0, T/\varepsilon^2]} \|g^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)} \\ & \leq C\varepsilon^{-1} \sup_{t \in [0, T/\varepsilon^2]} \{ \|\partial_t g^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)} + \|\nabla g^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)} \} . \end{aligned} \quad (3.20)$$

*Proof.* We first consider  $n \geq 2$ . Given  $\varepsilon > 0$ , we choose a tiling of the space as

$$\mathbb{R}^n = \bigcup_{m \in \mathbb{Z}^n} E_m^\varepsilon, \quad E_m^\varepsilon = x_m + [0, \varepsilon^{-1})^n, \quad x_m = m\varepsilon^{-1}. \quad (3.21)$$

Given the function  $g^\varepsilon$  we define a piecewise constant function through an averaging procedure,

$$\bar{g}^\varepsilon(x, t) := \int_{E_m^\varepsilon} g^\varepsilon(\xi, t) d\xi \quad \text{if } x \in E_m^\varepsilon. \quad (3.22)$$

The Poincaré inequality for functions with vanishing average allows to estimate

$$\begin{aligned} \|g^\varepsilon(\cdot, t) - \bar{g}^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2 &= \sum_m \|g^\varepsilon(\cdot, t) - \bar{g}^\varepsilon(\cdot, t)\|_{L^2(E_m^\varepsilon)}^2 \\ &\leq C \text{diam}(E_m^\varepsilon)^2 \sum_m \|\nabla g^\varepsilon(\cdot, t)\|_{L^2(E_m^\varepsilon)}^2 \leq C\varepsilon^{-2} \|\nabla g^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

This provides estimate (3.20) for the part  $g^\varepsilon - \bar{g}^\varepsilon$ .

In order to estimate  $\bar{g}^\varepsilon$ , we use the fact that averaging does not increase the  $L^2$ -norm,

$$\sum_m |E_m^\varepsilon| |\partial_t \bar{g}^\varepsilon(x_m, t)|^2 = \|\partial_t \bar{g}^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2 \leq \|\partial_t g^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2.$$

With the fundamental theorem of calculus we find

$$\begin{aligned} \|\bar{g}^\varepsilon(\cdot, t)\|_{L^\infty(\mathbb{R}^n)}^2 &= \max_m |\bar{g}^\varepsilon(x_m, t)|^2 \leq \max_m \frac{T^2}{\varepsilon^4} \sup_{s \in [0, T\varepsilon^{-2}]} |\partial_t \bar{g}^\varepsilon(x_m, s)|^2 \\ &\leq \frac{T^2}{\varepsilon^4} |E_m^\varepsilon|^{-1} \sup_{s \in [0, T\varepsilon^{-2}]} \sum_m |E_m^\varepsilon| |\partial_t \bar{g}^\varepsilon(x_m, s)|^2 \\ &\leq T^2 \varepsilon^{n-4} \sup_{s \in [0, T\varepsilon^{-2}]} \|\partial_t g^\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

For  $n \geq 2$ , this provides estimate (3.20) for the remaining part  $\bar{g}^\varepsilon$ .

In the case  $n = 1$  we proceed in a similar way, using now a tiling with pieces of larger diameter,

$$\mathbb{R} = \bigcup_{m \in \mathbb{Z}} E_m^\varepsilon, \quad E_m^\varepsilon = x_m + [0, \varepsilon^{-2}), \quad x_m = m\varepsilon^{-2}. \quad (3.23)$$

The estimate for  $\bar{g}^\varepsilon \in L^\infty(0, T\varepsilon^{-2}; L^\infty(\mathbb{R}^n))$  is obtained as above with the  $\varepsilon$ -factor  $\varepsilon^{-4}|E_m^\varepsilon|^{-1} = \varepsilon^{-2}$  as desired. To estimate the difference  $g^\varepsilon - \bar{g}^\varepsilon$  we use, in the case  $n = 1$ , the same  $L^\infty$ -based norm. We calculate, for arbitrary  $t \in (0, T\varepsilon^{-2})$ ,

$$\begin{aligned} \|g^\varepsilon(\cdot, t) - \bar{g}^\varepsilon(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} &= \sup_m \|g^\varepsilon(\cdot, t) - \bar{g}^\varepsilon(\cdot, t)\|_{L^\infty(E_m^\varepsilon)} \\ &\leq \sup_m \|\partial_x g^\varepsilon(\cdot, t)\|_{L^1(E_m^\varepsilon)} \leq \sup_m \text{diam}(E_m^\varepsilon)^{1/2} \|\partial_x g^\varepsilon(\cdot, t)\|_{L^2(E_m^\varepsilon)}. \end{aligned}$$

Because of  $\text{diam}(E_m^\varepsilon)^{1/2} = \varepsilon^{-1}$ , this shows (3.20). We emphasize that we obtain a pure  $L^\infty$ -bound on the left hand side of (3.20) in the case  $n = 1$ .  $\square$

## 4 Numerical results

In order to illustrate the approximation result of Theorem 1.1, we numerically solve equations (1.1) and (1.5) in dimensions  $n = 1$  and  $n = 2$  with the initial conditions in (1.2).

For the spatial discretization of (1.1) we choose the fourth order finite difference scheme of [6]. In one dimension ( $n = 1$ ) and for smooth  $a^\varepsilon(x)$  the value of  $\partial_x(a^\varepsilon(x)\partial_x u)$  at the grid point  $x = x_j$  is approximated by

$$(\mathbf{A}^\varepsilon(\lambda)u)_j := \frac{4}{3\Delta x} \left\{ a_{j+\frac{1}{2}}^\varepsilon \frac{u_{j+1} - u_j}{\Delta x} - a_{j-\frac{1}{2}}^\varepsilon \frac{u_j - u_{j-1}}{\Delta x} \right\} \quad (4.1)$$

$$- \frac{1}{6\Delta x} \left\{ a_{j+1}^\varepsilon \frac{u_{j+2} - u_j}{2\Delta x} - a_{j-1}^\varepsilon \frac{u_j - u_{j-2}}{2\Delta x} \right\}, \quad (4.2)$$

where the coefficients  $a_j^\varepsilon$  and  $a_{j+\frac{1}{2}}^\varepsilon$  are defined via  $a_j^\varepsilon = \frac{1}{2\Delta x} \int_{x_{j-1}}^{x_{j+1}} a^\varepsilon(x) dx$  and  $a_{j+\frac{1}{2}}^\varepsilon = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} a^\varepsilon(x) dx$ , and where  $\Delta x$  is the spacing of the uniform grid  $(x_j)_j$ . For the time discretization we use the standard centered second order scheme resulting in the fully discrete problem

$$u_j^{m+1} = 2u_j^m - u_j^{m-1} + (\Delta t)^2 (\mathbf{A}^\varepsilon(\lambda) u^m)_j.$$

In order to initialize the scheme, we set  $u_j^0 = f(x_j)$  and approximate  $u^1$  via the Taylor expansion  $u^1 = u^0 + \frac{(\Delta t)^2}{2} \mathbf{A}^\varepsilon(\lambda) u^0$ . For the evaluation of  $\mathbf{A}^\varepsilon(\lambda) u$  at the boundary of the computational domain we assume  $u = 0$  outside the domain. This is legitimate as we choose a large enough computational domain so that the solution is essentially zero at the boundary.

The effective equation (1.5) is solved via a second order centered finite difference scheme. For the second derivatives we use the standard stencil  $(\mathbf{D}_2 w)_j := (\Delta x)^{-2} (w_{j+1} - 2w_j + w_{j-1})$  and for the fourth derivatives we use  $(\mathbf{D}_4 w)_j := (\Delta x)^{-4} (w_{j+2} - 4w_{j+1} + 6w_j - 4w_{j-1} + w_{j-2})$  so that the semidiscrete problem in the case  $n = 1$  reads

$$((\mathbf{I} - \varepsilon^2 E \mathbf{D}_2) \partial_t^2 u)_j = ((A \mathbf{D}_2 - \varepsilon^2 F \mathbf{D}_4) u)_j.$$

We recall that  $E$  and  $F$  are scalars when  $n = 1$ . Discretization in time is performed analogously to the case of equation (1.1).

The above described methods generalize to  $n \geq 2$  dimensions in a natural way, see [6] for equation (1.1) with  $n = 2$ .

In general the parameters  $a^*$ ,  $\alpha$ , and  $\beta$ , which determine the coefficients  $A$ ,  $E$  and  $F$  in the effective equation, need to be computed numerically. They can be computed by numerically differentiating the eigenvalue  $\mu_0$  as defined in (2.19).

## 4.1 One space dimension

We choose the material function  $a_Y(y) = 1.5 + 1.4 \cos(y)$  and the initial data  $f(x) = e^{-0.4x^2}$  and numerically investigate the quality of the approximation given by the effective equation. For the coefficients  $A = a^*$  and  $C = \alpha$  we find

$$a^* \approx 0.5385, \quad \alpha \approx -0.5853,$$

so that  $AD^2 = a^* \partial_x^2 \approx 0.5385 \partial_x^2$ ,  $ED^2 = -\frac{1}{a^*} C \partial_x^2 \approx 1.0869 \partial_x^2$ .

Equation (1.1) was solved with  $\Delta x = 2\pi\varepsilon/30$  and  $\Delta t = 0.008$  and (1.5) was solved with  $\Delta x \approx 2\pi/100$  and  $\Delta t = 0.005$ . In Fig. 1 we plot  $u^\varepsilon$  and  $w^\varepsilon$  for  $\varepsilon = 0.05$  at  $t = 400 = \varepsilon^{-2}$  and for  $\varepsilon = 0.1$  at  $t = 200 = 2\varepsilon^{-2}$ . We see that in both cases the main peak and the first few dispersive oscillations are well approximated by the effective model. In the latter case, i.e. with  $t$  relatively large for a given  $\varepsilon$ , a slight disagreement in the wavelength of the tail oscillations is visible. Fig. 1 additionally shows oscillations traveling faster than the main pulse. These oscillations are physically meaningful as their speed is below the maximal allowed propagation speed  $\hat{c} := |Y| \int_{\mathbb{R}} a_Y^{-1/2}(y) dy$ , see [17], marked by the vertical dotted line.

In Fig. 2 we study the convergence of the  $L^2(\mathbb{R})$ -error for the same material function and initial data as above. The error is computed at  $\varepsilon = 0.2, 0.1$  and  $0.05$



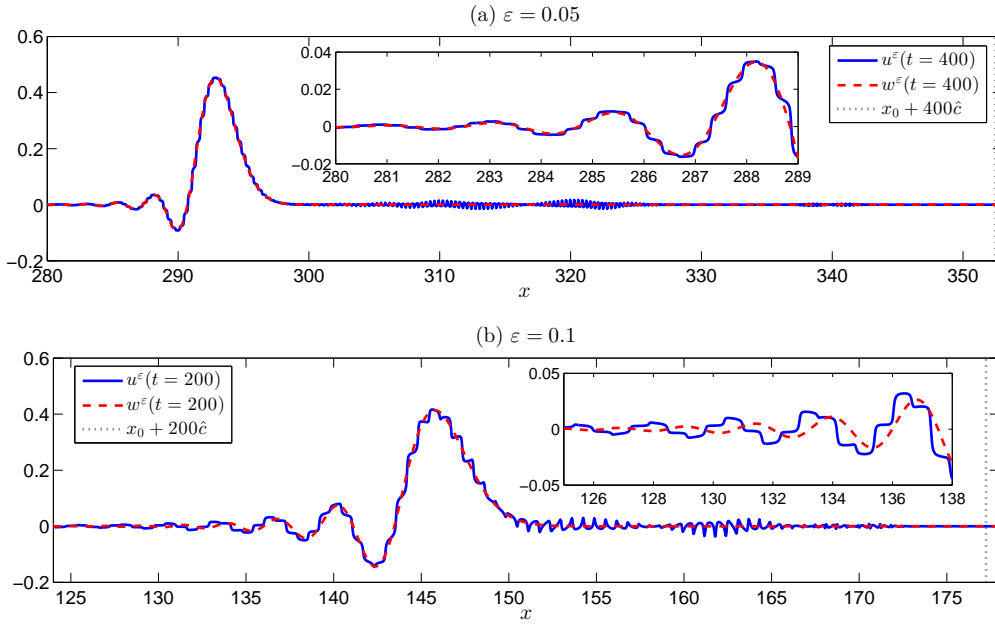


Figure 1: One-dimensional equation: the solutions  $u^\varepsilon$  and  $w^\varepsilon$  for  $a_Y(y) = 1.5 + 1.4 \cos(y)$  and  $f(x) = e^{-0.4x^2}$  are compared. Only the right propagating part of the solution is plotted. In (a) we have  $\varepsilon = 0.05$  and in (b)  $\varepsilon = 0.1$ . The insets zoom in on the dispersive oscillations to the left of the main peak.

and  $t = \varepsilon^{-2}$ . The error values are approximately 0.1954, 0.0977, 0.0494. Clearly, the numerical convergence is close to linear, in agreement with Theorem 1.1.

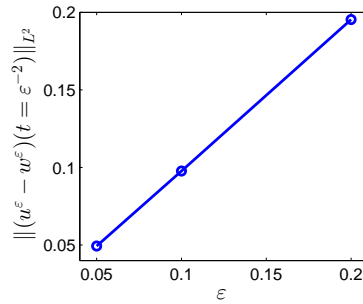


Figure 2: Convergence of the  $L^2$ -error  $\|u^\varepsilon - w^\varepsilon\|_{L^2}$  at  $t = \varepsilon^{-2}$  for  $a_Y(y) = 1.5 + 1.4 \cos(y)$ ,  $f(x) = e^{-0.4x^2}$ , and the three values  $\varepsilon = 0.2$ ,  $\varepsilon = 0.1$ , and  $\varepsilon = 0.05$ . We emphasize that this is a severe test for convergence: in both steps,  $\varepsilon$  is halved and the time instance is quadrupled.

## 4.2 Two space dimensions

Full two-dimensional ( $n = 2$ ) simulations for small values of  $\varepsilon > 0$  and time intervals of order  $O(\varepsilon^{-2})$  are computationally expensive due to the need to discretize

each period of size  $O(\varepsilon) \times O(\varepsilon)$  in a domain of size  $O(\varepsilon^{-2}) \times O(\varepsilon^{-2})$ . We therefore perform instead a simulation that is designed to mimic the long time behavior of a solution originating from localized initial data. After a long time the solution develops a large, close to circular, front. Within the strip

$$\Omega_s := x \in \mathbb{R} \times (-\varepsilon\pi, \varepsilon\pi)$$

we can expect that the front is nearly periodic in the  $x_2$ -direction. Therefore, we perform tests on  $\Omega_s$  with periodic boundary conditions in  $x_2$ , and initial data that are localized in  $x_1$  and constant in  $x_2$ . Our choice is to take  $f(x) = e^{-0.6x_1^2}$ ,  $x \in \Omega_s$ . We select a material function that describes a smoothed square structure, namely

$$a_Y(y) = 1 + c(y) - \bar{c}, \quad (4.3)$$

$$c(y) = \frac{1}{8} \prod_{j=1}^2 [1 + \tanh(4(y_j + \frac{3}{5}\pi))] [1 - \tanh(4(y_j - \frac{3}{5}\pi))],$$

where  $\bar{c} := \frac{1}{|Y|} \int_Y c(y) dy$ . This choice ensures a relatively large value of the dispersive coefficient  $\alpha$ . We find

$$a^* \approx 0.5808, \quad \alpha \approx -0.3078, \quad \beta \approx 0.0515.$$

These values correspond to case 2 in Remark 3.1 so that  $AD^2 = a^* \Delta \approx 0.5808 \Delta$ ,  $ED^2 = \frac{|\alpha|}{a^*} \Delta \approx 0.5300 \Delta$ ,  $FD^4 = (|\alpha| + 3\beta) \partial_{x_1}^2 \partial_{x_2}^2 \approx 0.4623 \partial_{x_1}^2 \partial_{x_2}^2$ . Due to the  $x_2$ -independence of the initial data, the solution of the effective model (1.5) on  $\Omega_s$  stays constant in  $x_2$  so that  $FD^4$  can be dropped and (1.5) becomes

$$\partial_t^2 w^\varepsilon = 0.5808 \partial_{x_1}^2 w^\varepsilon + \varepsilon^2 0.53 \partial_{x_1}^2 \partial_t^2 w^\varepsilon.$$

In the simulations of (1.1) we use  $\Delta x_1 = \Delta x_2 = 2\pi\varepsilon/30$  and  $\Delta t = 0.004$ , and in (1.5) we use  $\Delta x_1 = 2\pi/100$  and  $\Delta t = 0.01$ .

In Fig. 3 the main part of the right propagating half of the solution  $u^\varepsilon$  is plotted for  $\varepsilon = 0.1$  at  $t = 100 = \varepsilon^{-2}$ . One clearly sees dispersive oscillations behind the main pulse. Fig. 4 shows the agreement between  $w^\varepsilon$  and the  $x_2$ -mean of  $u^\varepsilon$  at  $\varepsilon = 0.1$  and  $t = \varepsilon^{-2}$ .

## Conclusions

We have performed an analysis of wave propagation in multi-dimensional heterogeneous media (periodic with length-scale  $\varepsilon > 0$ ). It is well-known that for large times, solutions cannot be approximated well by the homogenized wave equation. We have provided here a suitable well-posed dispersive wave equation of fourth order that describes the original solution  $u^\varepsilon$  on time intervals of order  $O(\varepsilon^{-2})$ . Our analytical results provide an error estimate of order  $O(\varepsilon)$  between  $u^\varepsilon$  and the solution  $w^\varepsilon$  of the dispersive equation. The coefficients of the effective equation are computable from the dispersion relation, which, in turn, is given by eigenvalues of a cell-problem. The qualitative agreement between  $u^\varepsilon$  and  $w^\varepsilon$  is confirmed by one-dimensional numerical tests, that even provide a confirmation of the linear convergence of the error in  $\varepsilon$ . In two space dimensions we can observe the validity of the dispersive equation in a simplified setting, computing solutions on a long strip.

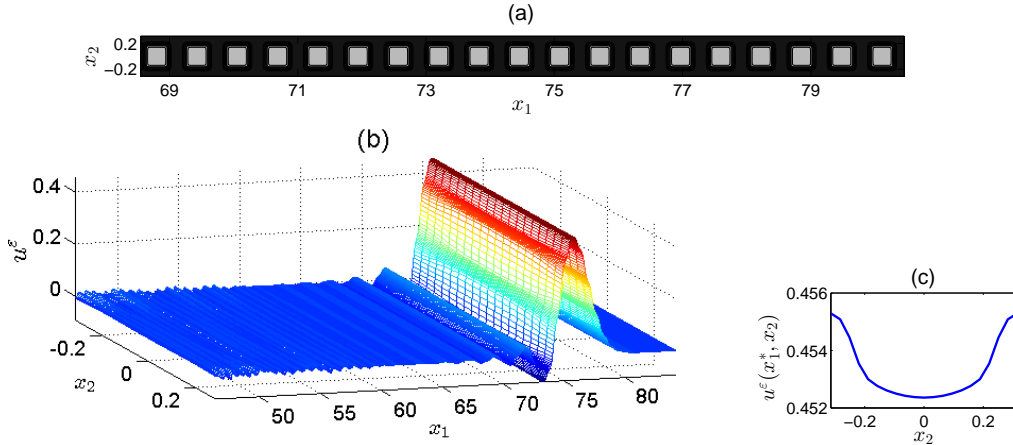


Figure 3: Two-dimensional equation: (a) The periodic structure  $a^\varepsilon(x)$  given by (4.3) over a section of the strip  $\Omega_s$ . (b) The main part of the right propagating part of the solution  $u^\varepsilon$  at  $t = 100$  for  $\varepsilon = 0.1$  and  $f(x) = e^{-0.6x_1^2}$ . (c) The  $x_2$ -profile of  $u^\varepsilon$  at  $x_1 = x_1^*$  with  $x_1^*$  being the position of the peak of the pulse.

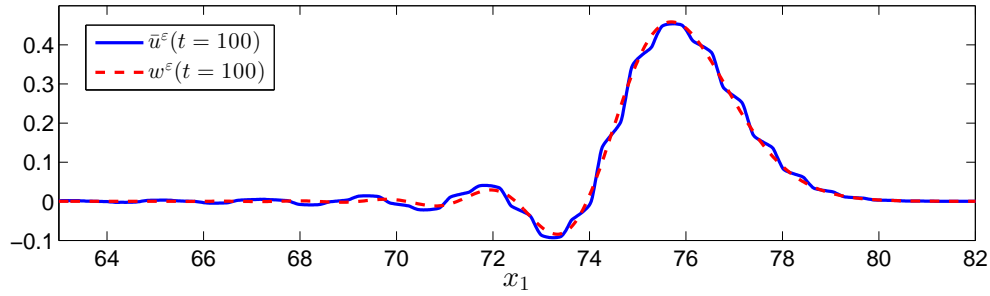


Figure 4: Comparison of  $\bar{u}^\varepsilon(x_1, t) := \frac{\varepsilon}{2\pi} \int_{-\varepsilon\pi}^{\varepsilon\pi} u^\varepsilon(x_1, x_2, t) dx_2$  and  $w^\varepsilon$  at  $\varepsilon = 0.1$  and  $t = \varepsilon^{-2}$  for  $a^\varepsilon(x)$  given by (4.3) and  $f(x) = e^{-0.6x^2}$ .

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